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# Leveller of Minds

If one should assume that a necessary condition to perfect thinking is that it shall be formulated *mathematically*, then, at once may be suggested a definition of mathematics somewhat as follows: *Mathematics is the science which perfects thought.*

Without pausing to justify the assumption—one with which many will take issue—it should yet not be out of order to face some of the consequences of a general adoption of the definition:

(1) It appears probable that fewer of the younger generation would brandish, without self-embarrassment, the declaration about themselves that they "have no head for mathematics, but plenty of ability in other fields."

(2) Closely linked with consequence (1) would be the increasing number of individuals who give serious attention to the imperfections in their thinking.

(3) The more or less popular conception (fostered considerably by some professional mathematicians) that the mathematical faculty is a special faculty, belonging only to specially constituted minds, should, largely, be shattered.

(4) Replacing the conception of mathematics described in (3) should be one which regards it as (a) a *meter* of intellectual power, (b) an instrument for developing the power.

In his now classic "Letters to a Young Man", Thomas DeQuincey, (1785-1859), one of the keenest of English thinkers, wrote:

> "In mathematics, upon two irresistible arguments which I shall set in a clear light when I come to explain the procedure of mind with regard to that sort of evidence, ...., there can be no *subtlety*. All minds are levelled except as to the rapidity of the course; and, from the entire absence of all those acts of mind which do really imply profundity of intellect, it is a question whether an idiot might not be made an excellent mathematician. Listen not to the romantic notions of the world on this subject! Above all, listen not to the mathematicians!"

We do not here undertake to separate truth from error in this DeQuincey deliverance. Never-the-less, measurably in agreement with the spirit of his opinion are two incontestable facts: (a) Every expanding field of pure knowledge uses an increasing body of mathematical formulation. (b) The schools of the world would, ultimately, be stripped of student programs in mathematics and the foundations of science and the higher mathematics would dissolve, if the doctrine were generally adopted that relatively few minds can acquire mathematics.

S. T. Sanders.

# Graphical Solutions for Complex Roots of Quadratics, Cubics and Quartics

By George A. Yanosik
*New York University*

While it is very well known that the real roots of an algebraic equation $f(x)=0$ may be readily found graphically by measurement of the $x$-intercepts of the graph of the left member of the equation, it is not widely known that certain simple measurements on the graph, together with simple arithmetic calculations, will also quickly give the complex roots. This paper will present such methods for quadratics, for cubics and for quartics having two real roots. It will be pre-supposed that the left member $f(x)$ was in reduced form when the graph was constructed.

1. *The Quadratic.* The complex roots being $a \pm ib$, the quadratic function may be denoted by

$$y=f(x)=x^2-2ax+a^2+b^2.$$

The graph will be a vertical parabola, whose equation can be put in the form

$$(x-a)^2=y-b^2,$$

from which it is evident that, if $OA$ and $AP$ are measured (Figure 1), the complex roots will be given by

$$a \pm ib = OA \pm i\sqrt{AP}$$

2. *The Cubic.* A method for obtaining the complex roots of a cubic from its graph has already been given* by the writer. This method will be briefly stated here without proof.

If the roots are $a$ and $b \pm ic$, the cubic function will be denoted by

$$y=f(x)=(x-a)(x^2-2bx+b^2+c^2).$$

Let a line be drawn through the point $A \equiv (a,0)$ tangent to the graph (Figure 2). If now the abscissa $BP$ of the point of tangency $P$ be measured, and also the slope $m(=V/H)$ of the tangent line, the complex roots will be given by

$$b \pm ic = BP \pm i\sqrt{m}.$$

*National Mathematics Magazine, January, 1936.

3. *The Quartic.* If the roots are $a$, $b$ and $c \pm id$, the quartic function is

$$y = f(x) = (x-a)(x-b)(x^2-2cx+c^2+d^2).$$

It will be shown that, if the bitangent $PQ$ be drawn (Figure 3), and the line $CZ$ be drawn through $C$, the midpoint of $AB$, parallel to $OY$, the complex roots of the quartic will be given by

$$c \pm id = (OC+DP+EQ) \pm i\sqrt{2 \cdot DP \cdot EQ + (CB)^2}.$$

For, if $CZ$ is used as a new $Y$-axis, the quartic function will be

$$y = F(x) = (x^2-e^2)(x^2-2fx+f^2+d^2),$$

in which $e = CB$ and $f = c - OC$.

Now if $y = mx+n$ is the bitangent, the equation

$$x^4-2fx^3+(f^2+d^2-e^2)x^2+(2e^2f-m)x-e^2(f^2+d^2)-n=0$$

must have two pairs of double roots, viz. $DP$, $DP$, $EQ$ and $EQ$. Hence $DP+EQ=f$, and $(DP)^2+4 \cdot DP \cdot EQ+(EQ)^2 = f^2+d^2-e^2$, so that

$$DP+EQ=f \quad \text{and} \quad 2 \cdot DP \cdot EQ = d^2-(CB)^2.$$

We now have that

$$c \pm id = (OC+DP+EQ) \pm i\sqrt{2 \cdot DP \cdot EQ + (CB)^2}.$$

The complex roots of a quartic with two complex roots can be shown to be closely linked with a certain vertical parabola. Consider the graph (Figure 4) of the quartic function

$$y = F(x) = (x^2-e^2)(x^2-2fx+f^2+d^2), \quad e \neq 0.$$

Among the infinitely many parabolas that might be drawn through the points $A$ and $B$, with equation $y = k(x^2-e^2)$, there is one which will have a common tangent (at $Q$) with the quartic. This parabola can easily be shown to be $y = d^2(x^2-e^2)$, and point $Q$ can be shown to have $x=f$ for its abscissa. Since $1/d^2$ is the latus rectum of this parbola, the complex roots can be obtained by

$$f \pm id = (\textit{abscissa of } Q) \pm i/\sqrt{\textit{latus rectum}}.$$

The complex roots can be found in still another way. The tangent drawn at $P$ can be shown to have a slope $m = 2e^2f$. Also $PO = e^2(f^2+d^2)$, so that

$$f \pm id = (m/2e^2) \pm i\sqrt{(PO/e^2)-(m^2/4e^4)}.$$

If the quartic has two real equal roots, the method just given cannot be used. The quartic is now given by

$$y = F(x) = x^2(x^2 - 2fx + f^2 + d^2)$$

and the vertical parabola by

$$y = d^2x^2.$$

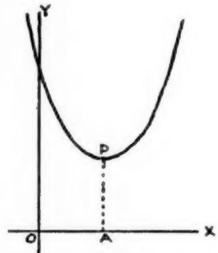


FIG. 1.

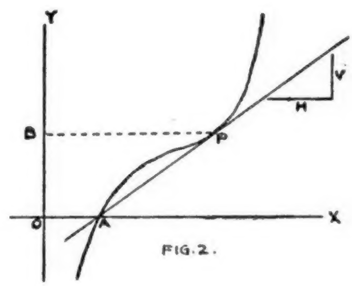


FIG. 2.

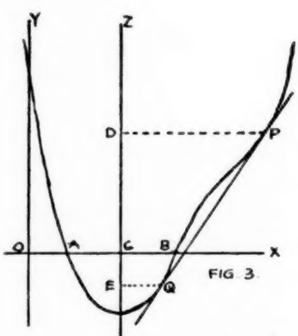


FIG. 3

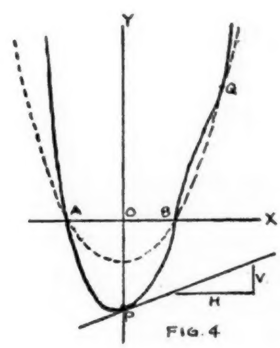


FIG. 4

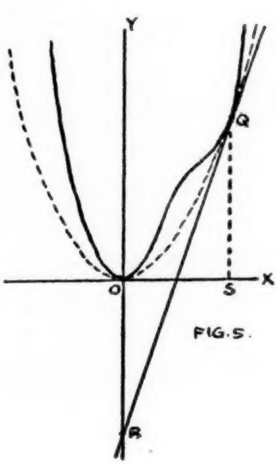


FIG. 5.

By a well-known property of the parabola, a tangent drawn at $Q$ must have a $y$-intercept $RO$ equal to the ordinate $SQ$ (Figure 5). Hence, to find the complex roots of a quartic with two real equal roots, a tangent line to the quartic must be drawn, by trial, so that distance $RO$ equals distance $SQ$. Now the quartic equation gives that

$$SQ = (OS)^2 d^2,$$

so that the complex roots are given by

$$f \pm id = OS \pm i\sqrt{QS}/OS.$$

# Necessary and Sufficient Conditions That Regression Systems of Sums with Elements in Common Be Linear

By Dzung-shu Wei
*The University of Iowa*

1. *Introduction.* Let repeated samples of $n$ independent items, $x_j$, be drawn from a population characterized by the continuous distribution function $f(x)$ of a chance variable $x$. To $k \angle n$ items chosen at random from each of these samples, add $m-k$ items, $x_j'$, taken at random from the population. Form the sums

$$y_1 = \sum_{j=1}^{k} x_j + \sum_{j=k+1}^{n} x_j$$

and

$$z_1 = \sum_{j=1}^{k} x_j + \sum_{j=k+1}^{m} x_j'.$$

That there is, in general, correlation between $y_1$ and $z_1$ due to common elements was probably first known to Kapteyn[4]. Various aspects of correlation attributed to common elements have been studied by other writers[3,6].

In a recent paper, Kenny[5] has proved by means of the characteristic functions that the regression systems of sums $y_1$ and $z_1$ are linear. If, however, a real constant is assigned to each element in the sums, the regression is not necessarily linear. Let these sums be

$$y = \sum_{j=1}^{k} a_j x_j + \sum_{j=k+1}^{n} a_j x_j \tag{1.1}$$

and

$$z = \sum_{j=1}^{k} c_j x_j + \sum_{j=k+1}^{m} c_j x'_j. \tag{1.2}$$

It is the purpose of this paper to determine necessary and sufficient conditions that such regression systems be linear.

Wicksell[8,9] has set up a criterion to test the linearity of the regression of two correlated variables. Allen[1] has likewise established

a necessary and sufficient condition for the linearity of the regression of sums

$$y_2 = a\xi + \alpha$$

and

$$z_2 = c\xi + \beta,$$

in which $\alpha$, $\beta$ and $\xi$ are mutually independent chance variables and $a$ and $c$ any real constants. It will be seen that the results we are to obtain agree with those of these authors.

2. *Preliminary Consideration.* If $F(y,z)$ is the joint distribution function of $y$ and $z$ and if $g(y)$ is the marginal distribution function of $y$, then for an assigned value of $y$, the mean value $\bar{z}_y$ of $z$ is given by

$$\bar{z}_y g(y) = \int_{-\infty}^{\infty} zF(y,z)dz. \tag{2.1}$$

It is known that if $\bar{z}_y$ is a linear function of $y$,

$$\bar{z}_y - {}_z\nu_1 = R_{zy}(y - {}_y\nu_1), \tag{2.2}$$

where $R_{zy}$ is the coefficient of regression of $z$ on $y$, and ${}_y\nu_1$ and ${}_z\nu_1$ are respectively the mean values of $y$ and $z$. It is to be understood that the values of $R_{zy}$, ${}_y\nu_1$ and ${}_z\nu_1$ exist. In terms of the constants $a_j$ and $c_j$ and the mean value $\nu_1$ of $x$, we have

$${}_y\nu_1 = \nu_1 \sum_{j=1}^{n} a_j, \tag{2.3}$$

$${}_z\nu_1 = \nu_1 \sum_{j=1}^{m} c_j \tag{2.4}$$

and

$$R_{zy} = \frac{\sum_{j=1}^{k} a_j c_j}{\sum_{j=1}^{n} a_j^2}. \tag{2.5}$$

Hence,

$$\bar{z}_y - \nu_1 \sum_{j=1}^{m} c_j = \frac{\sum_{j=1}^{k} a_j c_j}{\sum_{j=1}^{n} a_j^2} \left[ y - \nu_1 \sum_{j=1}^{n} a_j \right] \tag{2.6}$$

3. *A Theorem.* Let

$$\varphi(a_j t_1) = \int_{-\infty}^{\infty} e^{ia_j t_1 x} f(x)dx. \tag{3.1}$$

We shall prove the following

*Theorem I.* If none of the coefficients of $x_j$ in the sum $y$ is zero, then necessary and sufficient conditions that the regression of $z$ on $y$ be linear are

$$(3.2)\qquad \frac{1}{\sum_{s=k+1}^{n} a_s^2} \sum_{s=k+1}^{n} [\log \varphi(a_s t_1) - i v_1 a_s t_1] = \frac{1}{a_j^2}[\log \varphi(a_j t_1) - i v_1 a_j t_1],$$
$$j = 1, 2, \cdots k.$$

4. *Proof of the Theorem.* We shall first show the necessity of the conditions. The characteristic function $\varphi(t_1,t_2)$ of $F(y,z)$ can be written as

$$\varphi(t_1,t_2) = \prod_{j=1}^{k} \int_{-\infty}^{\infty} e^{i(a_j t_1 + c_j t_2)x_j} f(x_j)dx_j \prod_{j=k+1}^{n} \int_{-\infty}^{\infty} e^{i a_j t_1 x_j} f(x_j) dx_j$$

$$\prod_{j=k+1}^{m} \int_{-\infty}^{\infty} e^{i c_j t_2 x_j'} f(x_j')dx_j' = \prod_{j=1}^{k} \varphi(a_j t_1 + c_j t_2) \prod_{j=k+1}^{n} \varphi(a_j t_1) \prod_{j=k+1}^{m} \varphi(c_j t_2).$$

It follows that

$$(4.1)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it_1 y + it_2 z} F(y,z)dydz = \prod_{j=1}^{k} \varphi(a_j t_1 + c_j t_2) \prod_{j=k+1}^{n} \varphi(a_j t_1)$$
$$\prod_{j=k+1}^{m} \varphi(c_j t_2).$$

On setting $t_2 = 0$, we obtain the characteristic function $\varphi(t_1)$ of $g(y)$

$$(4.2)\qquad \varphi(t_1) = \int_{-\infty}^{\infty} e^{it_1 y} g(y)dy = \prod_{j=1}^{n} \varphi(a_j t_1).$$

If we take the partial derivative of both members of (4.1) with respect to $t_2$ and evaluate the result at $t_2 = 0$, we have

$$(4.3)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ize^{it_1 y} F(y,z)dydz$$
$$= \varphi(t_1) \left[ \sum_{j=1}^{k} \frac{c_j}{a_j} \frac{d}{dt_1} \log \varphi(a_j t_1) + i v_1 \sum_{j=k+1}^{m} c_j \right].$$

On account of (2.1)

$$(4.4)\qquad \int_{-\infty}^{\infty} i\bar{z}_y e^{it_1 y} g(y)dy = \varphi(t_1)\left[\sum_{j=1}^{k} \frac{c_j}{a_j}\,\frac{d}{dt_1}\log \varphi(a_j t_1) + i\nu_1 \sum_{j=k+1}^{m} c_j\right]$$

Suppose $\bar{z}_y$ is a linear function of $y$. Then by (2.7), we get, after some simplification,

$$(4.5)\qquad R_{zy}\left[\int_{-\infty}^{\infty} iye^{it_1 y} g(y)dy - i\varphi(t_1)\nu_1 \sum_{j=1}^{n} a_j\right]$$

$$= \varphi(t_1)\sum_{j=1}^{k}\left\{\frac{c_j}{a_j}\left[\frac{d}{dt_1}\log \varphi(a_j t_1) - i\nu_1 a_j\right]\right\}.$$

But the integral in (4.5) is

$$\varphi(t_1)\sum_{j=1}^{n}\frac{d}{dt_1}\log \varphi(a_j t_1).$$

Since $\varphi(t_1)$ never vanishes, we may divide out $\varphi(t_1)$ and obtain

$$(4.6)\qquad R_{zy}\sum_{j=1}^{n}\left[\frac{d}{dt_1}\log \varphi(a_j t_1) - i\nu_1 a_j\right]$$

$$= \sum_{j=1}^{k}\left\{\frac{c_j}{a_j}\left[\frac{d}{dt_1}\log \varphi(a_j t_1) - i\nu_1 a_j\right]\right\}.$$

We next integrate (4.6) on $t_1$, obtaining

$$(4.7)\qquad R_{zy}\sum_{j=1}^{n}\left[\log \varphi(a_j t_1) - i\nu_1 a_j t_1\right]$$

$$= \sum_{j=1}^{k}\left\{\frac{c_j}{a_j}\left[\log \varphi(a_j t_1) - i\nu_1 a_j t_1\right]\right\},$$

the constant of integration being zero.

Change $R_{zy}$ to its explicit expression. Since the constants $c_j$'s are arbitrary, (4.7) may be regarded as an identity in the $c_j$'s. Upon equating the coefficients of like $c_j$'s, we obtain, after dividing out $a_j$,

$$(4.8)\qquad \frac{1}{\sum_{s=1}^{n} a_s^2}\sum_{s=1}^{n}\left[\log \varphi(a_s t_1) - i\nu_1 a_s t_1\right] = \frac{1}{a_j^2}\left[\log \varphi(a_j t_1) - i\nu_1 a_j t_1\right],$$

$$j = 1, 2, \cdots k.$$

By virtue of an algebraic theorem on ratios, we reduce (4.8) to the form (3.2). Thus (3.2) gives the necessary conditions that the regression of $z$ on $y$ be linear.

We shall next inquire if these conditions are likewise sufficient. To this end, let us consider (4.4). By the Fourier transform, we have

$$\text{(4.9)}\quad i\bar{z}_y g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it_1y}\varphi(t_1)\sum_{j=1}^{k}\frac{c_j}{a_j}\frac{d}{dt_1}\log\varphi(a_jt_1)dt_1$$

$$+i\nu_1 g(y)\sum_{j=k+1}^{m} c_j.$$

From (3.2), we deduce (4.6) by retracing through the steps (4.8) and (4.7). When substituted in (4.9), we make use of (4.2) and (2.3) and obtain

$$\text{(4.10)}\quad \sum_{j=1}^{k}\frac{c_j}{a_j}\frac{d}{dt_1}\log\varphi(a_jt_1)$$

$$=R_{zy}\left[\frac{d}{dt_1}\log\varphi(t_1) - i_y\nu_1\right] + i\nu_1\sum_{j=1}^{k}a_j.$$

On account of (4.10) and (2.4), it follows that

$$\text{(4.11)}\quad ig(y)(\bar{z}_y - {}_z\nu_1) = \frac{R_{zy}}{2\pi}\int_{-\infty}^{\infty} e^{-it_1y}\varphi(t_1)\frac{d}{dt_1}\log\varphi(t_1)dt_1 - ig(y)\,{}_y\nu_1.$$

But $$\frac{R_{zy}}{2\pi}\int_{-\infty}^{\infty} e^{-it_1y}\varphi(t_1)\frac{d}{dt_1}\log\varphi(t_1)dt_1$$

$$=\frac{R_{zy}}{2\pi}\int_{-\infty}^{\infty} e^{-it_1y}\frac{d}{dt_1}\varphi(t_1)dt_1.$$

Upon integrating by parts, we get

$$\text{(4.12)}\quad \frac{R_{zy}}{2\pi}\int_{-\infty}^{\infty} e^{-it_1y}\frac{d}{dt_1}\varphi(t_1)dt_1$$

$$=\frac{R_{zy}}{2\pi}\left[e^{-it_1y}\varphi(t_1)\right]_{t_1=-\infty}^{t_1=\infty} + i\,yg(y).$$

The first term in the right-hand member vanishes uniformly as $|t_1|$ becomes infinite.[7] Hence, for an assigned value of $y$.

$$\bar{z}_y - {}_z\nu_1 = R_{zy}(y - {}_y\nu_1).$$

The proof is thus complete.

In an analogous manner, necessary and sufficient conditions for the linearity of the regression of $y$ on $z$ may be formulated. It is to be noticed that no one of the constants $c_j$ is zero.

These conditions are equally valid when the domain of variation of the chance variable $x$ is limited to an isolated set of values.

5. *Inferences.* Suppose we further assume that the moment $\nu_k$ of $f(x)$ exists for all values of $k=1, 2, \cdots$. Then $\log \varphi(a_j t_1)$ may be expanded into a power series in $t_1$. We shall now procure a clearer view of the types of distribution functions that satisfy the conditions, although the generality is weakened. With such an expansion, (3.4) becomes

$$(5.1)\qquad \frac{1}{\sum_{s=k+1}^{n} a_s^2} \sum_{s=k+1}^{n} \left[ \sum_{\gamma=1}^{\infty} \frac{(ia_s t_1)^\gamma}{\gamma!} \lambda_\gamma - i\nu_1 a_s t_1 \right]$$

$$= \frac{1}{a_j^2} \left[ \sum_{\gamma=1}^{\infty} \frac{(ia_j t_1)^\gamma}{\gamma!} \lambda_\gamma - i\nu_1 a_j t_1 \right],$$

$$j = 1, 2, \cdots k,$$

where the $\lambda_\gamma$'s are the semi-invariants[2] of $x$. We observe that (5.1) is independent of $\lambda_1$ and $\lambda_2$. Being an identity in $t_1$, the coefficients of like powers of $t_1$ must be equal. Consequently,

$$(5.2)\qquad \lambda_\gamma \sum_{s=k+1}^{n} \{ a_s^2 [ a_s{}^{\gamma-2} - a_j{}^{\gamma-2} ] \} = 0,$$

$$j = 1, 2, \cdots k,$$

$$\gamma = 3, 4, \cdots .$$

It is to be noted that (5.2) can be transformed into the conditions devised by Wicksell.

By applying the factor theorem of algebra, we derive the following inferences.

(i) If $\lambda_\gamma = 0$ for $\gamma = 3, 4, \cdots$, that is, if $f(x)$ is normal, $\bar{z}_y - {}_z\nu_1 = R_{zy}(y - {}_y\nu_1)$ for every set of non-zero constants $a_j$'s.

(ii) If $\lambda_\gamma = 0$ for all odd $\gamma$'s, that is, if $f(x)$ is symmetric, $\bar{z}_y - {}_z\nu_1 = R_{zy}(y - {}_y\nu_1)$ when $a_1 = a_2 = \cdots = a_k = a$, $a_{k+1} = a_{k+2} = \cdots a_n = b$ and $a = \pm b$.

(iii) For any type of distribution function, $\bar{z}_y - {}_z\nu_1 = R_{zy}(y - {}_y\nu_1)$, when the $a_j$'s are all equal. Kenney's results incidently fall in this class.

6. *A Generalization.* Suppose the sums of $y$ and $z$ are composed of the powers of the chance variable $x$, that is,

$$y_3 = \sum_{j=1}^{k} a_j x_j^p + \sum_{j=k+1}^{n} a_j x_j^q \tag{6.1}$$

and

$$z_3 = \sum_{j=1}^{k} c_j x_j^p + \sum_{j=k+1}^{m} c_j x_j^{\prime r}, \tag{6.2}$$

where $p$, $q$ and $r$ are positive integers and the $a_j$'s and the $c_j$'s are constants as before. Let

$$\varphi(a_j t_1, p) = \int_{-\infty}^{\infty} e^{i a_j t_1 x^p} f(x) dx. \tag{6.3}$$

Then we have

*Theorem II.* If none of the coefficients of $x_j$ in the sum $y_3$ is zero, and if the moments of required orders exist, then necessary and sufficient conditions that the regression system of $z_3$ on $y_3$ be linear are

$$\frac{\sum_{s=k+1}^{n}[\log \varphi(a_s t_1, q) - i\nu_q a_s t_1]}{(\nu_{2q} - \nu_q^2)\sum_{s=k+1}^{n} a_s^2} = \frac{\log \varphi(a_j t_1, p) - i\nu_p a_j t_1}{(\nu_{2p} - \nu_p^2) a_j^2}, \quad j = 1, 2, \cdots k. \tag{6.4}$$

The previous theorem is a special case of this one with $p = q = r = 1$. The argument is the same and the proof can be supplied without particular difficulty. It is of interest to observe the following case.

(iv) Suppose $f(x)$ is normal. If $p$ is unity and $q$ any other positive integer, or vice versa, it is readily seen that $\bar{z}_y$ is not a linear function of $y$. For there is at least one even $\gamma_0$ such that $\lambda_{\gamma_0}^{(xq)} \neq 0$. (for definition of $\lambda_{\gamma_0}^{(xq)}$ see).[2] When substituted in (6.5) we get

$$(\nu_2 - \nu_1^2)\lambda_{\gamma_0}^{(xq)} \sum_{s=k+1}^{n} a_s^{\gamma_0} = 0$$

or
$$\sum_{s=k+1}^{n} a_s \gamma_0 = 0.$$

But this is impossible, whence $\bar{z}_y$ is not a linear function of $y$. This result agrees with that of Allen.

## *BIBLIOGRAPHY*

[1] Allen, H. V., "A Theorem Concerning the Linearity of Regression", Statistical Research Memoir, Vol. II (1938) pp. 60-68.

[2] Craig, C. C., "An Application of Thiele's Semi-invariants to the Sampling Problem", Metron, Vol. 7, No. 4, (1928) pp. 3-74.

[3] Fischer, C. H., "On Correlation Surfaces of Sums with a Certain Number of Random Elements in Common", Annals of Mathematical Statistics, Vol. 4 (1933) pp. 103-126.

[4] Kapteyn, J. C., "Definition of Correlation Coefficient", Monthly Notices of the Royal Astronomical Society, Vol. 72 (1911-12) pp. 518-525.

[5] Kenney, J. F., "The regression Systems of Two Sums Having Random Elements in Common", Annals of Mathematical Statistics, Vol. 10 (1939) pp. 70-73.

[6] Rietz, H. L., "A Simple Non-normal Correlation Surface", Biometrika, Vol. 24 (1932) pp. 288-290.

[7] Titchmarsh, E. C., "Theory of Fourier Integrals", Oxford, (1937) p. 11.

[8] Wicksell, S. D., "Remarks on Regression", Annals of Mathematical Statistics, Vol. I (1930) pp. 3-13.

[9] Wicksell, S. D., "On Correlation Functions of Type III", Biometrika, Vol. 25 (1933) pp. 121-133.

# Geometric Examples of Convergent Series

By C. A. Barnhart
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The student of geometry is impressed by the frequency with which a set of magnitudes form a geometric progression. This may explain the origin of the term "geometric progression." While the majority of such geometric progressions consist of only three distinct magnitudes of which one is the geometric mean of the other two, there are some which are infinite geometric sequences. It is the purpose of this paper to present two interesting examples of convergent geometric sequences.

The basis for one arrangement of geometric magnitudes in a convergent sequence is furnished by the consideration of the problem of constructing a triangle, given its medians. This construction depends upon the following closely related theorems. Given a triangle $T_0$ and its first median triangle $T_1$, then the area of $T_1$ is three-fourths of the area of $T_0$. Similarly, if the second median triangle $T_2$ of $T_0$ be constructed of the medians of $T_1$, then the sides of $T_2$ are each equal to three-fourths of the corresponding sides of $T_0$. The fact that $T_0$ and $T_2$ are similar, and $T_1$, $T_3$ are also similar, suggests a graphical arrangement of a given triangle $T_0$ and its consecutive median triangles which will illustrate a convergent geometric sequence.

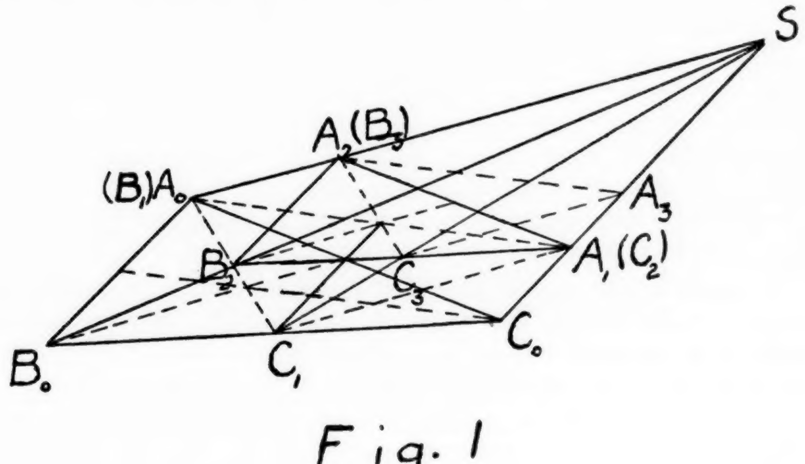


Fig. 1

Let us denote the sides of any triangle $T_i$ of the sequence $T_0, T_1, T_2, T_3 \cdots$ as $a_i, b_i, c_i$, and let us assume that $a_i, b_i, c_i$ are always arranged in counter-clockwise order so as to retain $a_i$ of $T_i$ in its position as a median of its preceding triangle $T_{i-1}$ in the sequence.

If $T_i$ then is constructed in agreement with this assumption, it is known that $a_i, b_i, c_i$ will be parallel respectively to the sides $a_{i-2}$, $b_{i-2}, c_{i-2}$ of $T_{i-2}$. Since the similar triangles $T_i$ and $T_{i-2}$ have their corresponding sides parallel, they are homothetic. Thus it is possible to obtain two sequences of homothetic triangles; one sequence for even values of $i$ and the other for odd values of $i$.

If the homothetic center of the sequence of homothetic triangles $T_0, T_2, T_4, \cdots$, be denoted by $S$, and the homothetic center of the sequence of homothetic triangles $T_1, T_3, T_5, \cdots$, be denoted by $S'$, then the homothetic rays $AA_2$, $BB_2$, $CC_2$ of the sequence of even-ordered median triangles pass respectively through $A_2, A_4, \cdots, B_2, B_4, \cdots$, $C_2, C_4, \cdots$; while the homothetic rays $A_1A_3$, $B_1B_3$, $C_1C_3$ of the sequence of odd-ordered median triangles pass respectively through $A_3, A_5, \cdots$, $B_3, B_5, \cdots, C_3, C_5, \cdots$.

But due to our assumption of counter-clockwise order in all the consecutive median triangles, and the retention of $a_i$ in its original position as a median of $T_{i-1}$, $B_1$ coincides with $A_0$, and the ray $B_1S$ coincides with the ray $A_0S$. Furthermore, since the lines joining corresponding points of homothetic polygons are all concurrent at the homothetic center; the ray $C_1S'$ which passes through the mid-points of the sides $a$ of the even-ordered triangles must pass through $S$. As a result $S'$ and $S$ coincide, and the ray $C_0S$ passes through $A_1, A_3, \cdots$, as well as through $C_2, C_4, \cdots$.

The homothetic center $S$ of both sequences of homothetic triangles indicates the convergence of each of the sequences. In fact, there are several divergent sequences illustrated in Fig. 1. In the case of the even-ordered median triangles with the three homothetic rays $A_0S$, $B_0S$, $C_0S$, we have the sequences $a_0, a_2, a_4, \cdots$; $b_0, b_2, b_4, \cdots$; $c_0, c_2, c_4, \cdots$; as well as all other sequences of corresponding line segments; for which the homothetic ratio, $r=3/4$, is less than 1.

Again in the case of the odd-ordered median triangles with the three homothetic rays $B_1S$ $(A_0S)$, $A_1S$ $(C_0S)$, $C_1S$, there are the corresponding sequences $a_1, a_3, a_5, \cdots$; $b_1, b_3, b_5, \cdots$; $c_1, c_3, c_5, \cdots$, as well as for all other sequences of corresponding line segments.

But using all four homothetic rays $A_0S$, $B_0S$, $C_0S$, $C_1S$, the areas of the entire sequence of consecutive triangles $K_0, K_1, K_2, K_3, \cdots$ is a convergent geometric sequence with the homothetic ratio $R'=3/4$.

By employing the same assumption, but using altitudes instead of medians, an arrangement of the given triangle $T_0$ and its consecutive altitude triangles may be obtained as another example of a convergent geometric sequence. For, if the cevians are altitudes instead of medians, it is known that the areas of the consecutive altitude triangles $T_{i-1}$, $T_i$ are related as follows:

$$\frac{K_i}{K_{i-1}} = \frac{2K_i}{2K_{i-1}} = \frac{a_i a_{i+1}}{a_{i-1} a_i} = \frac{b_i b_{i+1}}{b_{i-1} b_i} = \frac{c_i c_{i+1}}{c_{i-1} c_i}$$

$$= \frac{a_{i+1}}{a_{i-1}} = \frac{b_{i+1}}{b_{i-1}} = \frac{c_{i+1}}{c_{i-1}} = k \quad \text{(const.)}$$

Therefore the areas of the given triangle $T_0$ and its consecutive altitude triangles form a geometric sequence. This relationship also proves that $T_0, T_2, T_4, \cdots$, and $T_1, T_3, T_5, \cdots$, form two sequences of similar triangles.

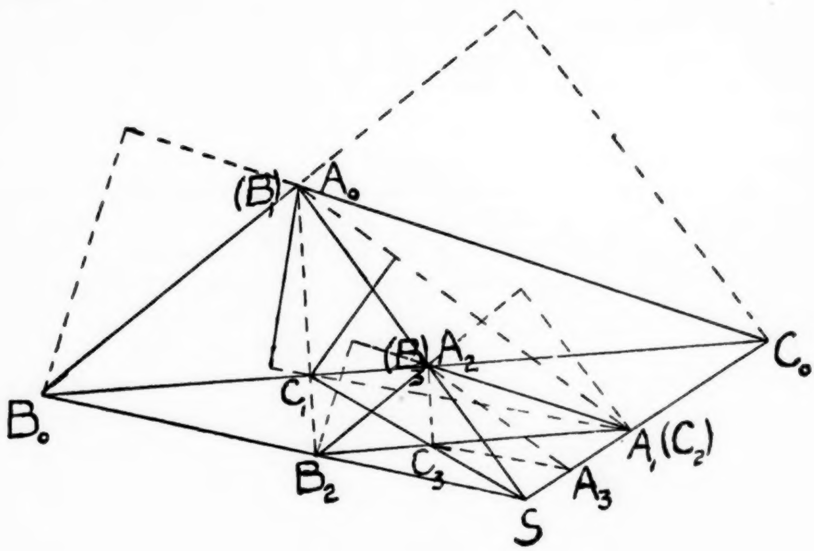


Fig. 2

Again by our assumption of counter-clockwise order, and the retention of $a_i$ in its original position as an altitude of $T_{i-1}$, $a_i$ is perpendicular to $a_{i-1}$, and $a_{i+1}$ is perpendicular to $a_i$. Therefore $a_{i+1}$ and $a_{i-1}$ are parallel, which proves that $T_0$ and all of its even-ordered

altitude triangles form one sequence of homothetic triangles, while the odd-ordered altitude triangles form another sequence of the same nature.

It is known, however, that the sum of the three altitudes of a triangle is less than the perimeter of the triangle so that the arbitrary constant

$$k=\frac{a_{i+1}}{a_{i-1}}=\frac{b_{i+1}}{b_{i-1}}=\frac{c_{i+1}}{c_{i-1}}=\frac{a_{i+1}+b_{i+1}+c_{i+1}}{a_{i-1}+b_{i-1}+c_{i-1}}$$

$$<\frac{a_i+b_i+c_i}{a_{i-1}+b_{i-1}+c_{i-1}}<\frac{a_{i-1}+b_{i-1}+c_{i-1}}{a_{i-1}+b_{i-1}+c_{i-1}}=1.$$

Therefore, a triangle and its consecutive altitude triangles can be arranged so as to give two convergent geometric sequences of homothetic triangles; one of the original triangle $T_0$ and its even-ordered altitude triangles, the other of its odd-ordered altitude triangles, so that in both cases all corresponding line segments form convergent geometric sequences. Finally the areas of $T_0$ and all of its consecutive altitude triangles form a third convergent geometric sequence. In all these sequences the homothetic ratio is the arbitrary constant $k<1$, and not an absolute constant as in the case of the triangle $T_0$ and its consecutive median triangles.

The arrangement of the consecutive cevian triangles in either case depends upon the constructibility of the first cevian triangle $T_1$, which is always constructible in the case of the medians, but is not not constructible in the case of the altitudes for some triangles with obtuse angles.

# A Locus Related to the Euler Line

By Karleton W. Crain
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Let the unit circle be cut by a line in two real distinct points, $A$ and $B$. These two points and any other point, $P$, on the circle determine a triangle the locus of whose circumcenter as $P$ moves around the circle is obviously the center of the given circle.

Let the unit circle have the equation

$$(1) \qquad x^2+y^2=1$$

and the given line the equation $y=k$ in rectangular coordinates. Let the moving point be $P(x_1,y_1)$. Then the coordinates of the orthocenter of the triangle $ABP$ are $x=x_1$ and

$$y=\frac{x_1^2-ky_1+2k^2-1}{k-y_1}\ .$$

Using the fact that $P$ is on the given unit circle we may eliminate $x_1$ and $y_1$ from these two equations as follows:

$$y(k-y_1)=(x_1^2-1)+k^2+k^2-ky_1 \quad \text{or,} \quad y(k-y_1)=k^2-y_1^2+k(k-y_1).$$

If $(k-y_1)\neq 0$, $y=2k+y_1$* or,

$$(2) \qquad x^2+y^2-4ky+4k^2-1=0.$$

Thus the locus of the orthocenters as $P$ moves about $x^2+y^2=1$ is the circle (2) whose radius is also unity and whose center is $(0,2k)$. It may be noted that $y=k$ is the radical axis of (1) and (2).

It is evident by homology that the locus of a point which divides $OR$ (where $O$ is the origin and $R$ is a point on (2)) into a constant ratio is also a circle. From the analytic point of view, the coordinates of such a point are $x=rx_2/(r+1)$ and $y=ry_2/(r+1)$ where $r$ is the ratio of division and $(x_2,y_2)$ any point on (2). On eliminating $x_2$ and $y_2$ from these equations and (2), we have

$$(3) \qquad (x^2+y^2)(r+1)^2-4kr(r+1)y+r^2(4k^2-1)=0.$$

The center of this circle is $(0,2kr/(r+1))$ and its radius $|r/(r+1)|$.

*Although $y_1=k$ when $P$ is at $A$ or $B$, the limiting position of the orthocenter is a real point. If $P$ is at $B$ the orthocenter is $(\sqrt{1-k^2},3k)$ which lies on the circle (2).

Since $OR$ is the Euler line of the triangle $ABP$, the locus of the centroids (when $r=1/2$ in equation (3)) is a circle whose radius is 1/3 and whose center is $(0,2k/3)$. If $r=1$, (3) is the locus of the Nine point Centers—a circle whose radius is 1/2 and whose center is $(0,k)$. (cf. Figure 1.)

It is interesting to note that (3) represents a circle with a real radius and real center whether or not the line $y=k$ intersects (1) in real points. Furthermore, the radius of (3) is independent of $k$.

Let us now consider the condition that will make $y=mx$ tangent to each circle of the family (3). After substituting $y=mx$ into (3), we have

$$x^2(1+m^2)(r+1)^2-4krm(r+1)x+r^2(4k^2-1)=0.$$

From this the required condition is

$$16k^2r^2m^2(r+1)^2-4r^2(4k^2-1)(1+m^2)(r+1)^2=0.$$

Since $r\neq -1$ in the finite plane, we have, if $r\neq 0$, $m=\pm\sqrt{4k^2-1}$. Since $m$ is real, $|k|\geqq 1/2$. This condition corresponds to the geometric necessity of keeping the origin from being inside the circle (2).

Let us now consider only the members of this family which form an unlimited chain of tangent circles. (cf. Figure 2.) Starting with the circle of equation (2) their radii, taken in decreasing order of magnitude, form a G. P. whose common ratio is $(|2k|-1)/(|2k|+1)$. Therefore, if $1/2<|k|<1$ we have a real, non-trivial case in which the sum of the radii is $(|2k|+1)/2$; the sum of the circumferences is $\pi(|2k|+1)$; the sum of the areas is $\pi(|2k|+1)^2/|8k|$.

# Note on the Quartic and Its Hessian*

By James A. Ward
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The object of this paper is to determine the relation between the roots of a quartic equation and those of its Hessian in the case where the Hessian is a cubic. The relation between the roots of a cubic and its Hessian and the application of this result to the solution of cubic equations has already been published by the author in the National Mathematics Magazine.†

Consider a quartic equation $Q(z)=0$ and let

$$f(x,y)=y^4Q\left(\frac{x}{y}\right).$$

Let $h(x,y)$ be the Hessian of $f(x,y)$ and let

$$H\left(\frac{x}{y}\right)=\frac{1}{y_4}h(x,y).$$

Then $H(z)=0$ is called the Hessian equation of $Q(z)=0$.

Theorem 1. If the roots of $Q(z)=0$ are increased by $\kappa$, the roots of $H(z)=0$ are also increased by $\kappa$.

This follows from the fact that the Hessian is a covariant of the form.

By increasing the roots by a suitable quantity, any quartic may be put into the reduced form.

$$z^4+pz^2+qz+r=0. \tag{1}$$

The Hessian of this is

$$8pz^4+24qz^3+(48r-4p^2)z^2-4pqz+(8pr-3q^2)=0. \tag{2}$$

The degree of (2) is less than 4 if and only if $p=0$. By theorem 1 this gives

*Read before the Southeastern Section of the Mathematical Association of America, March 30, 1940.

†"Using the Hessian to Solve the Cubic", Volume 9 (1935), 235-240.

Theorem 2. A necessary and sufficient condition that the Hessian of $Q(z)=0$ be of degree less than 4 is that the reduced equation of $Q(z)=0$ be of the form.

$$z^4+qz+r=0. \tag{3}$$

Let us leave aside as trivial the case $z^4+r=0$. Then the Hessian of (3) becomes

$$z^3+2rz^2/q-q/8=0. \tag{4}$$

Let its roots be $r_1$, $r_2$, $r_3$ then $q=8r_1r_2r_3$, $r=-4r_1r_2r_3(r_1+r_2+r_3)$. The Descartes Resolvent* of (3) is

$$64h^6+64r_1r_2r_3(r_1+r_2+r_3)h^2-64r_1{}^2r_2{}^2r_3{}^2=0$$

whose roots are $h^2=r_1r_2,\ r_1r_3,\ r_2r_3$. Hence, following Descartes' solution, one obtains

Theorem 3. The roots of a quartic with third and second degree terms lacking are

$$\begin{aligned} z_1 &= -\sqrt{r_1r_2}-\sqrt{r_1r_3}-\sqrt{r_2r_3} \\ z_2 &= -\sqrt{r_1r_2}+\sqrt{r_1r_3}+\sqrt{r_2r_3} \\ z_3 &= \quad\sqrt{r_1r_2}-\sqrt{r_1r_3}+\sqrt{r_2r_3} \\ z_4 &= +\sqrt{r_1r_2}+\sqrt{r_1r_3}-\sqrt{r_2r_3} \end{aligned}$$

where $r_1$, $r_2$, $r_3$ are the roots of the Hessian of the quartic and the square roots are chosen so that $\sqrt{r_1r_2}\,\sqrt{r_1r_3}\,\sqrt{r_2r_3}=q/8$.

This theorem may be used to solve numerical equations whose Hessians are cubics. Since by Theorem 2 every such equation can be reduced to one of type (3), let us consider

$$z^4+12z-5=0.$$

Its Hessian is $\qquad 6z^3-5z^2-9=0,$

from which it is seen that

$$r_1=3/2, \qquad r_2=\frac{-1+2i\sqrt{2}}{2}, \qquad r_3=\frac{-1-2i\sqrt{2}}{2},$$

$$\sqrt{r_1r_2}=\frac{\sqrt{2}+2i}{2}, \qquad \sqrt{r_1r_3}=\frac{\sqrt{2}-2i}{2}, \qquad \sqrt{r_2r_3}=1.$$

$$\therefore \quad z_1=-1-\sqrt{2}.$$

*Dickson, L. E., *First;Course in Theory of Equations*, Wiley, page 52.

$$z_2 = 1 - 2i$$

$$z_3 = 1 + 2i$$

$$z_4 = -1 + 2i.$$

By a quartic (Tschirnhausen) transformation* every quartic may be put into form (3). Hence Theorem 3 furnishes a theoretical solution of every quartic.

As an additional result, whether $Q(z) = 0$ is put into form (3) or not, we have

Theorem 4. If $r_1$ is a double root of $Q(z) = 0$, it is also a double root of $H(z) = 0$.

If a quartic has zero as a double root, its Hessian also has zero as a double root. The theorem then follows from Theorem 1.

*Dickson, L. E., *Modern Algebraic Theories*, Sanborn, page 212.

# An Algebraic Function of Geometric Figures

By Howard D. Grossman
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The following interesting connection between algebra and geometry suggests itself. Represent a geometric figure of $m$ dimensions containing only points, straight lines, planes, etc., by the algebraic function $f \equiv a_0 - a_1 d + a_2 d^2 - \cdots + (-1)^m d^m$, where $a_k$ is the number of $k$-dimensional bounding entities in the figure. The unit coefficient of $d^m$ (disregarding the sign) represents the one total figure.

Then a point $=1$; a straight line $=2-d$; a plane $n$-gon $= n - nd + d^2$; etc. Successive generation of square, cube, hypercube, etc. (or straight line topological equivalents) from a line, have the algebraic counterparts:

$$(2-d)^2 = 4 - 4d + d^2,$$

$$(2-d)^3 = 8 - 12d + 6d^2 - d^3,$$

$$(2-d)^4 = 16 - 32d + 24d^2 - 8d^3 + d^4, \quad \text{etc.}$$

All these coefficients are geometrically verifiable. In general, the geometric process whereby a straight line becomes a plane $n$-gon is represented by the algebraic operations $x[(n-2)-d]-n+4$, so that successive generations from a straight line $=2-d\,(=[1-(1-d)^2]/d)$ of triangle and triangular pyramid, for instance, are represented by:

$$(2-d)(1-d)+1 = 3 - 3d + d^2\,(=[1-(1-d)^3]/d)$$

$$(3-3d+d^2)(1-d)+1 = 4 - 6d + 4d^2 - d^3\,(=[1-(1-d)^4]/d).$$

Successive generations of a plane $n$-gon and $n$-gonal pyramid from a straight line are represented by:

$$(2-d)[(n-2)-d]-n+4 = n - nd + d^2,$$

$$(n-nd+d^2)(1-d)+1 = (n+1) - 2nd + (n+1)d^2 - d^3.$$

The generation of a prism from a plane $n$-gon is represented by:

$$(n-nd+d^2)(2-d) = 2n - 3nd + (n+2)d - d^3.$$

If we reverse all coefficients but the last in the algebraic function of a figure, we obtain the function of its *dual.*

In this language Euler's Law $V+F=E+2$ or $a_0-a_1+a_2-1=1$ becomes $f(1)=1$ for 3-dimensional figures. This law has analogs, viz. $a_0-1=1$ for 1 dimension, $a_0-a_1+1=1$ for 2 dimensions,

$$a_0-a_1+a_2-a_3+1=1$$

for 4 dimensions, etc., all of which are equivalent to $f(1)=1$. This equation is true for all the initial figures, and $f(1)$ is verifiably invariant under all the operations, here mentioned. E. g., if $f(1)=1$ and $g=f\times[(n-2)-d]-n+4$, then $g(1)=n-3-n+4=1$.

Perhaps this can be amplified: extended to higher dimensions, to more irregular polyhedra, to figures not bounded by planes like the cone, sphere, etc.; and applied to topological problems.

# *The Teachers' Department*

*Edited by*
JOSEPH SEIDLIN, JAMES MCGIFFERT, J. S. GEORGES
and L. J. ADAMS

## A Seminar Plan in Mathematics

By SISTER HELEN SULLIVAN, O. S. B., Ph.D.
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In an earlier issue of the NATIONAL MATHEMATICS MAGAZINE it was stated that "the teacher of college mathematics is primarily a teacher and secondarily a research mathematician". Later on in the same article we find "his (the college mathematics teacher) educational interests manifest themselves either in the contributions he makes to the formulation or solution of instructional problems, or by his reactions to the reported contributions of other teachers".* We agree whole-heartedly with both statements. In fact the writer's reaction to the second quotation in the foregoing was a strong desire to communicate to her professional colleagues the details of an instructional experiment which she has carried on at Mount Saint Scholastica College for women. It is hoped that the present article will stimulate comment, suggestions, and criticisms on the part of teachers similarly engaged.

The experiment deals with a course entitled *Reading List* which was introduced into the mathematics department in 1940. The technical course offerings are such as to provide a balanced training in algebra, geometry, analysis, and applied mathematics. The *Teachers' Course in Mathematics* acquaints the student with the pedagogical aspects of the subject. The course in *History of Mathematics* is, as its description indicates, intended "to show the historical development of mathematics from early Egyptian times to the 20th century".

It seemed to us that there was something missing from the program to develop the "at-home-ness" necessary for mastery in the field. Something had to be done to enable the student to build up a

*"Educational Interests of Teachers of College Mathematics", NATIONAL MATHEMATICS MAGAZINE, Vol. 16, p. 89.

solid structure of inter-related mathematical truths which would withstand the shock of the unknown. When the student's knowledge is used with the familiar and supple ease of personal ownership, then and then only can he be said to be a master of the field in an undergraduate sense.

Aside from our personal convictions in the matter, college authorities contend that those students who major in mathematics and the sciences possess only computational skills. They question the cultural, literary, and philosophic contributions of our plan of study. They point an accusing finger at the mathematics student who cannot (like his fellow classmen who pursue literature and fine arts) converse fluently regarding his courses. The reason for this they wrongly attribute to mathematics itself. Perhaps the correct explantion is found in the fact that we have been so busy in passing out piece-meal courses that we have not guided and assisted the student to correlate and organize his knowledge as befits a college senior.

The Catholic University of America has sponsored an intensive reading program in the undergraduate school for the past five years. Each student is given an *Upper Division Reading List* and is asked to make "systematic digests" of his readings under "organized headings" which the staff adviser examines at "undefined intervals".

The procedure employed at Mount Saint Scholastica College 1940-1941 is described in what follows. The course was open only to majors in the department. It was conducted on the seminar plan and might be more properly termed a pro-seminar than a formal class. About two weeks of intensive reading and discussion were devoted to each of the eight major units. Before undertaking the study of a new unit a list of *required* readings and suggested references was posted on the departmental bulletin board so that at the time of the semi-weekly meetings there was sufficient *common* matter to ensure discussion. The student response was most encouraging to the writer who secretly hopes it will be productive of some worth-while undergraduate research. The main purpose of the topic outline which was sufficiently elastic to take care of the immediate side-problems that arose was to furnish a general plan and to provide a coherent sub-structure. As individual interests deepened, each member was encouraged to undertake a project which would indicate her growth in and familiarity with some one phase of mathematics. A wide range of interests was manifested in these seminar papers which were presented and defended before the group. One student, with marked artistic ability, gave an excellent paper on *Dynamic Symmetry*. Another of a more disputatious temperament presented a scholarly treatise (carefully authenti-

cated) on the *Newton-Leibniz Controversy*. A third offered some new aspects regarding the relationship of mathematics to sociology. The other papers were primarily of an historic nature.

We feel that the students derived some real educational benefits from this project as the "comprehensive examinations" taken at the close of their senior year revealed a certain thoroughness which those of previous years did not. It is very probable that similar seminar programs are being carried on at other colleges and we shall welcome comments, suggestions, and additions to the outline.

## OUTLINE OF UNITS FOR INVESTIGATION

The Arabic numerals refer to the books listed in the general bibliography at the end of this section.

I. General Mathematical Background
(1) C. I; (2) Author's Preface, C. II, C. III; (3) C. IX, C. X; (4) C. I; (5) C. I; (6) C. I; (7) C. IV, C. V.

II. NUMBER
(1) C. II, C. III; (4) C. II, C. III; (8) C. III; (9) C. I; (10) C. I, C. II; (11) C. IV, C. V, C. IX; (6) Pp. 25-38, 56-68, C. XII.

III. GEOMETRY
(1) C. V, C. VI; (4) C. IV; (5) C. II; (12) C. IV; (8) C. IV.

IV. MODERN GEOMETRY
(1) C. VII; (5) C. III, C. IV, C. V; (10) pp. 326-330, 337-338, 389-403; (11) C. I, C. II.

V. FUNCTIONS
(1) C. VIII, C. IX; (13) C. II; (14) pp. 302-304; (5) C. VI, C. VII, C. VIII, C. IX; (8) C. XI; (11) C. VI; (6) pp. 129-138; C. XXI; (15) C. II, C. VI.

VI. APPLIED MATHEMATICS
(1) C. XI, C. XII, C. XIII; (8) C. XII; (5) C. X, C. XI, C. XII; (16) Chaps. I-IX; (7) C. VII.

VII. LIMITS
(1) C. XIV, C. XV, C. XVI; (10) p. 613, p. 619; (5) C. XIII, C. XIV, C. XV.

VIII. NON-EUCLIDEAN GEOMETRY
(1) C. XVIII; (4) C. V; (17) Chaps. I-III; (6) C. XV; (5) C. XVI; (10) pp. 351-388; (11) C. III.

## GENERAL BIBLIOGRAPHY

(1) *Introduction to Mathematics:* H. R. Cooley, D. Gans, M. Kline, H. E. Wahlert.
(2) *Philosophy of Science:* F. J. Sheen.
(3) *Outposts of Science:* B. Jaffe.
(4) *Teaching of Secondary Mathematics:* J. P. Hassler and R. R. Smith.

(5) *Men of Mathematics:* E. T. Bell.
(6) *Development of Mathematics:* E. T. Bell.
(7) *Mathematics in Modern Education:* 11th Yearbook N. C. T. M.
(8) *Mathematics for the Million:* L. Hogben.
(9) *The Number Concept:* L. L. Conant.
(10) *Source Book in Mathematics:* D. E. Smith.
(11) *Modern Mathematics—Monographs:* J. W. A. Young.
(12) *A Short Account of History of Mathematics:* W. W. R. Ball.
(13) *Course in Pure Mathematics:* G. H. Hardy.
(14) *An Invitation to Mathematics:* A. Dresden.
(15) *Functional Thinking:* Ninth Yearbook N. C. T. M.
(16) *Mathematics in Modern Life:* Sixth Yearbook N. C. T. M.
(17) *Non-Euclidean Geometry:* D. M. Y. Sommerville.

# Is Mathematics an Exact Science?*

By Cecil B. Read
*University of Wichita, Wichita, Kansas*

Probably a teacher learns as much from his students as the student learns from his teacher. Some years ago when I was attempting to introduce the concept of negative and fractional exponents to a class in college algebra I had a student bring the text which he had studied in high school. He pointed out the statement: "The student will see that it is impossible for an exponent to be anything but a positive integer." I could find no qualifying statement. The student asked, "In the face of this contradiction how do you justify your statement that mathematics is an exact science?" Later in the same class I suggested to the students that they attempt to obtain a formula for the cost of shipping a package by parcel post if the charge is eight cents for the first pound and three cents for each additional pound. Most of the students gave a formula such as $c=8+(n-1)3$. With a little discussion the class soon saw that the formula failed if the number of pounds $n$ was, for example, $3\frac{1}{2}$. Likewise they saw that the graph of the cost function would be discontinuous. Again their high school texts were presented. Each gave the formula I have mentioned or an equivalent one. Two of the three gave a straight-line as the graph of the cost function. Again I was asked, "Is mathematics an exact science?"

According to the Oxford dictionary, exact sciences are those which admit of absolute precision, especially the mathematical sciences. If a science admits of absolute precision, it would seem that this preciseness would apply to definitions of terms. Yet it is not at all unusual for a teacher to discover that some of his better students can ask rather embarrassing questions. The questions may be more or less trivial, as for example, Why, in an exact science, do texts disagree in the labeling of axes in a three dimensional coordinate system? Of course this may

*Since mathematics is man-made (admitting that "God made the integers"), if mathematicians disagree or are careless, mathematics "loses out". It seems to us that when this war is over, some International Congress of Mathematicians might restore mathematics to its rightful position of the queen (or the equivalent of "queen" in a people's world) of exact sciences. We appreciate that this congress may need to tackle fearlessly such momentous and controversial issues as "tan $90=\infty$" vs. "tan 90 'just isn't'." Parenthetically we might add that "mathematician" is not synonymous with "mathematics textbook writer."—Ed.

be explained as the acceptance of one of two conventions (the student may still want to know why the alternate convention was not mentioned). Explanations are not so easy when one finds in two texts, supposed to be written for students on the same level of ability, statements which are directly contradictory, with no hint whatsoever that a different statement or definition may be found if one consults another authority. (This is by no means to be considered an argument for or against any particular definition, but rather a plea that where there is disagreement among authorities, the beginning student should at least have warning that such disagreement exists.)

With respect to the discrepancies, several shall be mentioned briefly. For example, a widely used text in calculus states that $n!$ has no meaning if $n$ is not a positive integer. Two pages later the student encounters 0! If he happens to use at least one text in advanced calculus he finds that factorials of fractional numbers are defined by means of the gamma function and he encounters such notations as $(-\frac{1}{2})!$

A slightly different situation arises when a term with a definite meaning in one field is used with a different meaning in another field. To quote a single example: a calculus text states, "The pressure on a surface of a given area submerged to a specified depth in a fluid is the weight of the column of the fluid which could be supported by the area." A reputable physics handbook gives as the definition of pressure: "force applied to, or distributed over a surface; measured as force per unit area." Apparently the calculus book is using the term total pressure as synonymous with total force. This discrepancy has been pointed out by many students in beginning calculus.

An example which illustrates incompleteness rather than discrepancy or contradiction is the definition found in many algebra text books:

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

An important restriction is the necessity that $a$ be positive if $n$ is an even integer. For example, what is the value of $(-32)^{2/10}$? Is involution to be performed first? If the exponent is written as 1/5 we have still not settled the question of the value of $(-32)^{3/10}$.

Let us pass to another illustration of inconsistency which, although it may not be of particular importance, again raises the question, "Is mathematics an exact science?" Is zero to be classified as an integer? Many texts are indefinite, one uses a definition specifically including zero as an integer, another specifically excludes zero. Again, is zero an imaginary number? Some texts specify that $a+bi$ is a pure imagi-

nary if and only if $a=0$ and $b\neq 0$. Others specifically state that zero is a pure imaginary. One author points out that zero is the only number common to the real and pure imaginary system.

Another example of incompleteness: many authors leave the distinct impression that the obtaining of parametric equations is a unique process. Of even more importance is the very common failure to point out that the graph given by the parametric equations may be only a portion of the graph given by the equation obtained by eliminating the parameter. As a simple example, compare the graph $x=\sin^2 t$, $y=\cos^2 t$, with the graph obtained by eliminating the parameter $t$.

In the field of statistics if a city of 100,000 increases its population 10% each year many books speak of a constant rate of change. In the sense of calculus this is definitely not a constant rate of change. A better term would be constant ratio of change.

In the field of trigonometry we find marked discrepancy in the definition of the principal values for the inverse cotangent, secant, and cosecant. According to some texts if $x$ is negative the principal value of arc csc $x$ falls in the interval from $-\pi/2$ to zero, according to others, in the interval from $-\pi$ to $-\pi/2$. The identical discrepancies are found in introductory texts in calculus.

Two additional examples might be mentioned, (1) Is the $x$ axis an asymptote of the curve $y=e^{-x}\sin x$ ? (2) Can the mantissa of a logarithm be a negative number? In both cases the answer is yes or no, depending on the reference consulted. With respect to the second question it may be noted that some texts define the mantissa as the decimal portion of a logarithm; others restrict the term to the situation where the logerithm is expressed as the sum of an integer and a decimal fraction which is positive or zero and less than unity. Under the restrictions stated, the positive fraction is the mantissa. If to obtain the common logarithm of one-half we subtract log 2 from log 1 (the result being $-.3010$) according to one text the mantissa is .3010, according to another it is .6990.

The list could be extended, but no doubt sufficient illustrations have been given at least to raise the question. The answer may be: Yes, mathematics is an exact science, but mathematicians use inexact terminology.

# Oblique Projection, Simplified

By Isidor F. Shapiro
*New York City*

Prompt visualization of objects as a whole is vital in technical work, both in school and in the plant. An Oblique Drawing enables the object to be readily envisioned, and has the advantage of being speedily executed, even by tyros. Usually we can quickly "box" the object, i. e., plot its features, freehand if desired, within the sectionally-latticed framework of a box, lightly sketched to contain it, utilizing proportions along edges and diagonals of sections of the box.

But when there are intricate details, and especially if they are to be accurately represented, it is much easier to utilize the following automatic method (this is an adaptation of the delightfully direct pathways laid down by Freese)[1]. The final spacious effect can be controlled by the convenient device of Fig. 3 below. The method is easily acquired in one or two exercises.

I. *Essentials.* (1) The first step is merely to ***obliquely*** project the picture onto *PP*3, from the specially placed *PP*1 and *PP*2 of Fig. 1: use one slope (***ad lib*** or *RO* of Fig. 3) for parallel oblique projection-lines ***downward*** from the points of *PP*1 to $FL_{1-3}$; and another slope (***ad lib*** or *TO* of Fig. 3) for projection-lines ***sideward*** from the points of *PP*2 to $FL_{2-3}$.

(2) We then complete in the standard way: drop a vertical from each such projection-point of $FL_{1-3}$ to intersect a horizontal from the correspondingly lettered projection-point of $FL_{2-3}$; the points that thus materialize on *PP*3 are joined by lines as indicated in the working drawings.

That is all there is to the procedure. But the best results call for purposeful selection of the ***ad lib*** obliques, as below.

II. *Refinements.* (a) *Spaciousness.* More space for detail may be desired on the top, say, than on the side of the final picture. This

[1] Ernest Irving Freese, in his "Perspective Projection" (The Pencil Points Press, New York, 1938) stressed the bias arrangement of *PP*1 and *PP*2. He did not apply his method to Oblique Projection, and so had no occasion to develop the convenient device for slopes shown in Fig. 3.

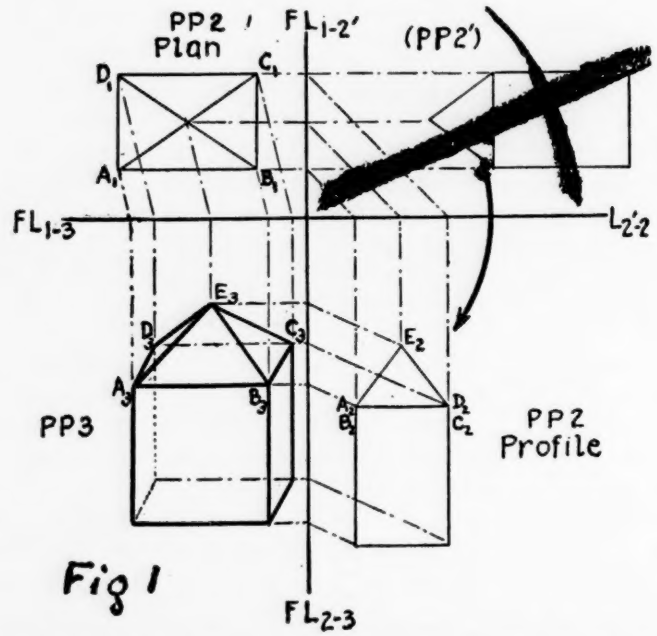


Fig 1

The Profile view is first executed (with its borders) upon paper tacked in the conventional site $PP2'$. That is then removed, *rotated*, and tacked or pasted in the site $PP2$, erect. The crossed-out zone $PP2'$ is thus left blank.

dictates making a freehand sketch of a basic cube[2] like Fig. 2. Evidently the determining factor is the $X$-axis, i. e., its foreshortened length and inclination in this sketch.

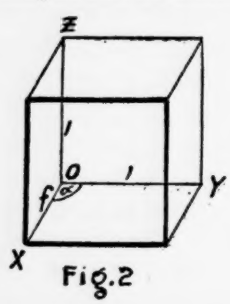


Fig.2

$OY$ (horizontal) $= OZ$ (vertical) $=$ unity.

$f =$ foreshortening fraction for $OX$ in the picture.

$\alpha =$ representation-angle (for picturing the objective angle of 90° that lies between the actual $OX$ and $OY$ axes).

(b) *Slopes*. Then the short graphic method of Fig. 3 suffices to determine the two oblique projection-slopes $RO$ and $TO$ as below

[2] The unit "basic cube" rests flat, with one whole face against the Picture Plane. Its axes are the "basic axes", and should be discriminated from any other rectangular formations which might also be present, but tilted, in the objective setting.

(which will automatically produce the intended spaciousness). This figure, though built upon Fig. 2, is introduced merely as a composite drawing device, and hence is most practically looked upon schematically, as a *plane* construction.

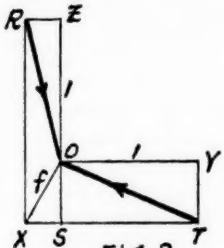


Fig. 3

$OY$ (horizontal) $= OZ$ (vertical) $=$ unity.

$OX = f$; $\angle XOY = \alpha$.

Extend vertical $ZO$ to meet in $S$ a horizontal from $X$; draw horizontal $ST = 1$.

Draw $XR$ equal and parallel to $SZ$.

$RO$ is the correct slope to use in Fig. 1, for projection-lines down from $PP1$ to $FL_{1-3}$;

$TO$ is the correct slope to use in Fig. 1, for projection-lines sideward from $PP2$ to $FL_{2-3}$.[3]

It will be observed that these slopes depend on the characteristics selected for $OX$, in Fig. 2.

When precise scale-work is desired, as for textbook illustrations, this arrowed gnomon should be executed as a working memorandum in the blank space $PP2'$ of our Fig. 1 layout; and there scale statements, for unity and $f$, may be notated. This device then serves two purposes: (1) initially it guides the draftsman as to the proper projection-slopes; and (2) at the end of the work it specifies the "basic axial" directions along which scale measurements can be directly read off the picture.

(c) *Remarks.* No precision is required in selecting angle $\alpha$ (its reproduction in the picture is automatic when slopes $RO$ and $TO$ are used for projection). But our choice of $f$ should decidedly be some convenient fraction for scale-reading, such as 1/2, or 5/6, etc., or unity, or even longer.[4]

Time can be further saved whenever the simpler proportional method becomes convenient for subdividing main features that have been outlined by projection-lines.

[3] The proof becomes apparent if we utilize in Fig. 1 the elementary working drawings of a "basic cube", and draw its Oblique Projection in $PP3$ per a pair of slopes. By translating those slopes we derive Fig. 3.

[4] Students are apt to overlook the fact that in Oblique Projection a picture-line can be longer than its corresponding objective-line, if the latter be not parallel to the *PP*, and if the slope of the actual projection-rays is acute enough to the *PP*.

# *Problem Department*

*Edited by*
E. P. STARKE and N. A. COURT*

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

## *SOLUTIONS*

No. 460. Proposed by *William N. Huff*, The Hill School, Pottstown, Pennsylvania.

On each edge of a regular octahedron choose one point such that the two opposite edges are divided in the ratio $a/b$ and the other edges in the ratio $b/a$. Find $a/b$ so that these points are vertices of a regular icosahedron.

Solution by the *Proposer*.

Let $L, M, N, O, P, R$ be the six vertices with $R, M, O$ opposite $L, P, N$, respectively. Let $B$ and $C$ be points dividing $LM$ and $LP$ respectively, in the ratio $a/b$; and let $A$ and $D$ divide $LO$ and $LN$ in the ratio $b/a$. If the length of each edge of the octahedron is $a+b$, we have

(1) $$LB = LC = OA = ND = a, \qquad LA = LD = MB = PC = b.$$

In the triangle $LAC$ the length of $AC(=AB=DB=DC)$ is given by

$$\overline{AC}^2 = \overline{LA}^2 + \overline{LC}^2 - 2\cdot\overline{LA}\cdot\overline{LC}\cos 60^\circ = \overline{LA}^2 + \overline{LC}^2 - \overline{LA}\cdot\overline{LC},$$

or with the use of (1),

$$\overline{AC}^2 = b^2 + a^2 - ab.$$

In the triangle $LMP$, $BC$ is parallel to $MP$, and we have $LB : LM = BC : MP$ which reduces to

$$BC = a\sqrt{2}.$$

Finally, triangle $ABC$ is equilateral if $AC=BC$ or

$$a^2+b^2-ab=(a\sqrt{2})^2,$$

whence (using the positive root)

$$a/b=\tfrac{1}{2}(\sqrt{5}-1).$$

No. 469. Proposed by "*Troubled*".

Find the locus of the center of a circle which touches any pair of elements selected from fixed points, fixed lines, and fixed circles.

Solution by *David L. MacKay*, Evander Childs High School, New York City.

Denoting the variable center by $X$, and fixed point, line and circle, respectively by $P$, $L$ and $C$ (having center $C$ and radius $r$), we have the following cases:

1. $(P_1, P_2)$. The locus is the perpendicular bisector of $P_1P_2$.

2. $(L_1, L_2)$. According as the fixed lines are intersecting or parallel, the locus consists of the bisectors of the angles formed by $L_1$ and $L_2$ or the line midway between $L_1$ and $L_2$.

3. $(L, P)$. Since the distances of $X$ from $L$ and $P$ are equal, the locus of $X$ is a parabola having $P$ as focus and $L$ as directrix.

4. $(C, L)$. Since the distance of $X$ from $L$ equals $XC\pm r$, $X$ is equidistant from $C$ and one of the lines parallel to $L$ and at a distance of $r$ from $L$. Thus the locus of $X$ consists of two parabolas with common focus at $C$ and with the lines parallel to $L$ for directrices. One parabola corresponds to circles tangent externally to the fixed circle $C$, the other parabola to circles tangent internally. (In case $C$ is tangent to $L$, one of the lines parallel to $L$ passes through $C$ so that the corresponding parabola degenerates into the line perpendicular to $L$ at $C$. A similar remark applies in case 3 above if $L$ goes through $P$.)

5. $(C_1, C_2)$. Unless one circle lies wholly within the other, we evidently have

$$|XC_1-XC_2| = |r_1\pm r_2).$$

Since the right member is constant, the locus consists of two hyperbolas with foci at $C_1$ and $C_2$ and with transverse axis $2a=|r_1\pm r_2|$. Centers of circles tangent internally to one of the fixed circles and externally to the other lie on the hyperbola for which $2a=r_1+r_2$; circles tangent internally to both fixed circles have their centers on

one branch of the hyperbola for which $2a=|r_1-r_2|$, and the other branch corresponds to circles tangent externally to both $C_1$ and $C_2$.

If one fixed circle lies wholly within the other, we have

$$XC_1+XC_2=|r_1\pm r_2|,$$

whence the locus of $X$ consists of two ellipses whose foci are $C_1$ and $C_2$ and for which $2a$ equals $r_1+r_2$ or $|r_1-r_2|$. These values of the major axis correspond respectively to circles tangent internally to one of the fixed circles and externally to the other and to circles tangent internally to both fixed circles.

If $C_1$ and $C_2$ are tangent, one of the hyperbolas (or ellipses) degenerates into the common normal at the point of tangency.

6. $(C, P)$. If in the above we set $r_2=0$, we have the present case. The hyperbolas (or ellipses) there mentioned become coincident, and the desired locus is therefore a hyperbola (or ellipse) with foci at $C$ and $P$. If $P$ lies on $C$, the locus of $X$ is the diameter of $C$ which passes through $P$.

Also solved by *Paul D. Thomas.*

No. 472. Proposed by *Paul D. Thomas,* Lucedale, Miss.

If $a$, $b$, $c$, $d$ are the lengths of the sides of a quadrilateral which admits both a circumscribed and inscribed circle, show that the in-radius is

$$r=\frac{\sqrt{abcd}}{s},$$

where $2s$ is the perimeter.

Solution by *D. L. MacKay,* Evander Childs High School, New York City.

The area of an inscriptible quadrilateral is given by

$$\Delta=\sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{1}$$

If a circle can also be inscribed in the quadrilateral, the relations

$$a+c=b+d, \qquad \Delta=rs$$

must hold so that (1) becomes

$$rs=\sqrt{abcd}$$

as required.

Also solved by the *Proposer,* and by *F. A. Lewis* who finds the problem in Loney, *Trigonometry,* (second edition), p. 256.

No. 474. Proposed by *W. V. Parker*, Louisiana State University.

There is one, and only one, triangle of maximum area inscribed to the ellipse $x^2/a^2+y^2/b^2=1$, with vertex at any point $P$ of the ellipse. Find the locus of the circumcenter of this triangle as $P$ moves around the ellipse.

Solution by *Paul D. Thomas*, U. S. Coast and Geodetic Survey, Sherburne, N. Y.

When a triangle inscribed in an ellipse is a maximum, the eccentric angles of its angular points are $r$, $r+2\pi/3$, $r+4\pi/3$.*

The coordinates of the center of the circle through the points whose eccentric angles are $r_i(i=1,2,3)$ are given by†

$$x=\frac{a^2-b^2}{4a}\left[\sum\cos r_i+\cos\left(\sum r_i\right)\right],\quad y=\frac{b^2-a^2}{4b}\left[\sum\sin r_i-\sin\left(\sum r_i\right)\right].$$

These two relations give

$$x=\frac{a^2-b^2}{4a}\cos 3r \qquad y=\frac{a^2-b^2}{4b}\sin 3r.$$

Elimination of the parameter $r$ leads to the equation of the required locus

$$a^2x^2+b^2y^2=(a^2-b^2)^2/16,$$

an ellipse similar to the original ellipse.

Also solved by the *Proposer*.

No. 475. Proposed by *E. P. Starke*, Rutgers University.

The triangle formed by three tangents to the ellipse, $x=a\cos\Theta$, $y=b\sin\Theta$, has the area

$$S=ab\tan\frac{\Theta_2-\Theta_1}{2}\tan\frac{\Theta_3-\Theta_2}{2}\tan\frac{\Theta_1-\Theta_3}{2},$$

where $0\leqq\Theta_1<\Theta_2<\Theta_3<2\pi$ are the values of the parameter for the three points of contact. The value of $S$ is positive or negative according as the ellipse is inscribed in the triangle or not. The formula for the area of the triangle inscribed in the ellipse with vertices at these points of contact is

$$2ab\sin\frac{\Theta_2-\Theta_1}{2}\sin\frac{\Theta_3-\Theta_2}{2}\sin\frac{\Theta_3-\Theta_1}{2},$$

*C. Smith, *Conic Sections*, p. 173.

†*Ibid.*, p. 169.

Solution by *Paul D. Thomas*, U. S. Coast and Geodetic Survery, Sherburne, N. Y.

The area of the triangle determined by the three points

$$P_i(a \cos \Theta_i, \quad b \sin \Theta_i), \quad i=1,2,3,$$

is given by the determinant

$$(1) \quad K=\tfrac{1}{2}ab|\cos \Theta_i, \sin \Theta_1, 1|$$

$$=\tfrac{1}{2}ab[\sin(\Theta_2-\Theta_1)-\sin(\Theta_3-\Theta_1)+\sin(\Theta_3-\Theta_2)].$$

The trigonometric identity

$$(2) \qquad \sin A+\sin B-\sin(A+B)=4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}(A+B),$$

easily established, reduces the expression in the brackets of (1) to $4 \sin \frac{1}{2}(\Theta_3-\Theta_2) \sin \frac{1}{2}(\Theta_2-\Theta_1) \sin \frac{1}{2}(\Theta_3-\Theta_1)$, so that $K$ becomes

$$K=2ab \sin \tfrac{1}{2}(\Theta_2-\Theta_1) \sin \tfrac{1}{2}(\Theta_3-\Theta_2) \sin \tfrac{1}{2}(\Theta_3-\Theta_1).$$

The equations of the tangents at points $P_i$ are

$$\frac{x}{a} \cos \Theta_i+\frac{y}{b} \sin \Theta_i=1, \qquad i=2,2,3.$$

These meet in the points

$$\left(a \cos\frac{\Theta_k+\Theta_j}{2} \Big/ \cos\frac{\Theta_k-\Theta_1}{2}, \qquad b \sin\frac{\Theta_k+\Theta_j}{2} \Big/ \cos\frac{\Theta_k-\Theta_j}{2}\right),$$

where $k,j=2,1;\ 3,1;\ 3,2$. The area of the triangle formed by these three points is

$$S=\frac{ab}{2}\left|\cos \frac{\Theta_k+\Theta_j}{2}, \quad \sin \frac{\Theta_k+\Theta_j}{2}, \quad \cos\frac{\Theta_k-\Theta_j}{2}\right| \Big/ \Pi \cos\frac{\Theta_k-\Theta_j}{2}.$$

Expanding the determinant according to elements of the last column one finds the value of $S$ in the form

$$S=\frac{ab}{4} \frac{[\sin(\Theta_3-\Theta_2)-\sin(\Theta_3-\Theta_1)+\sin(\Theta_2-\Theta_1)]}{\cos\frac{\Theta_3-\Theta_2}{2} \cos \frac{\Theta_3-\Theta_1}{2} \cos \frac{\Theta_2-\Theta_1}{2}}.$$

The identity (2) applied to this numerator produces at once

$$S=ab \tan \frac{\Theta_2-\Theta_1}{2} \tan \frac{\Theta_3-\Theta_2}{2} \tan \frac{\Theta_1-\Theta_3}{2}.$$

These problems are found in C. Smith, *Conic Sections*, pp. 162, 178.

## PROPOSALS

No. 498. Proposed by *F. C. Gentry*, Louisiana Polytechnic Institute.

Prove
$$\begin{vmatrix} \cos B\cos C & \cos A\cos C & \cos A\cos B \\ \cos A & \cos B & \cos C \\ bc & ac & ab \end{vmatrix} = 0,$$

where, as usual, the letters represent angles and sides of any triangle.

No. 499. Proposed by *F. C. Gentry*, Louisiana Polytechnic Institute.

If $A$, $B$, $C$ are the angles of a triangle show that
$$2+2\cos A\cos B\cos C=\sin^2 A+\sin^2 B+\sin^2 C$$

and hence show that the determinant
$$\begin{vmatrix} 0 & y+x\cos C & z+x\cos B \\ x+y\cos C & 0 & z+y\cos A \\ x+z\cos B & y+z\cos A & 0 \end{vmatrix}$$
$$=(x\sin A+y\sin B+z\sin C)(yz\sin A+xz\sin B+xy\sin C),$$

identically in $x$, $y$, and $z$.

No. 500. Proposed by *Robert C. Yates*, West Point, N. Y.

Discuss the variation of the angle of elevation of a gun (fixed muzzle velocity) such that a barrage may be laid down along a specified path. (Assume a plane terrain.)

No. 501. Proposed by *Howard D. Grossman*, New York City.

If $n$ is a positive integer, prove the following:
$$\frac{a^{2n+1}+b^{2n+1}}{a+b}=\sum_{j=0}^{n}\binom{n+j}{2j}(a-b)^{2j}(ab)^{n-j},$$
$$\frac{a^{2n}-b^{2n}}{a+b}=\sum_{j=0}^{n}\binom{n+j}{2j+1}(a-b)^{2j+1}(ab)^{n-j-1}.$$

No. 502. Proposed by *E. P. Starke*, Rutgers University.

A wagers that B, in making $k$ successive tosses with a coin will never have a run of four or more heads. Find the values of $k$ for which the odds are against A.

No. 503. Proposed by *William E. Taylor*, Student, Colgate University.

$A$ and $B$, with abscissas $a$ and $b$, respectively, $(0<a<b)$, are points of the curve $x^2y=1$. Consider the rectangle having sides parallel to the coordinate axes and with $A$ and $B$ for a pair of opposite vertices. Prove that the curve divides the area of this rectangle in the ratio $b:a$.

# *Bibliography and Reviews*

*Edited by*
H. A. SIMMONS and P. K. SMITH

*The Gist of Mathematics.* By Justin H. Moore and Julio A. Mira, Prentice-Hall, New York, 1942. viii+726 pages. $5.35.

This book of 31 chapters is devoted to an attempt to overcome the "widely prevailing timidity" with which many people face mathematics. The authors have defined the title to mean the elementary concepts which economists, biologists, and scientific workers in all fields find necessary.

The subject matter is that of the college preparatory courses in algebra, plane and solid geometry, introductory trigonometry, with applications to the theory of conic sections and derivatives of polynomials. In addition to the problems at the end of each chapter, pages 583-687 contain additional exercises and problems. Appendices A (Greek alphabet), B (Mensuration formulas), C (theorems of geometry), are followed by an index.

The *Gist of Mathematics* might well be placed in every high school library for use as a parallel or reference text. Doubtless some of the unusual chapter headings such as, "*The Bugbear, Modern Wizardry, Figure Skating in the Sky,*" would serve to intrigue the reader into finding out what it is all about.

*Virginia Military Institute.* W. E. BYRNE.

*Algebra.* (The first of a series: *Mathematics for Technical Training*). By P. L. Evans. Ginn and Company, New York, 1942. viii+123 pages.

The range of material covered in this little book is something of an invitation to persons who may wish to get the working essentials of algebra, but have not the time (or courage) for a larger course. The usual topics of intermediate algebra, through quadratic equations, are covered in the first hundred pages.

The explanations are necessarily brief, and the worked examples are not numerous. Very good lists of exercises are available in each chapter. In a good many cases, the principles and procedures are stated as rules instead of being developed as logical preferences or necessities.

Persons who have enough interest and scientific aptitude to succeed in technical work may be expected to cover this text with a fair measure of success. However, students who depend on the text and the teacher to explain all points requiring thought may be expected to find the work difficult.

In certain sections, the diction is not clear, and the authority of definition is not good. For example, (p. 2) "*To add two numbers having unlike signs,* subtract the smaller absolute value from the larger and give the result to sign of the larger absolute value". Since absolute values do not have signs, the author must have meant: ... *give the result the sign of the number having the larger absolute value.*

The statement (p. 9) "An algebraic expression with two or more terms is called a ***polynomial***" is not in keeping with the better usage of the term polynomial.

On the subject of factoring, (p. 22), we quote "A trinomial of the form $(x^4+kx^2y^2+y^4)$ can be reduced to the difference of two squares if $k$ is such that the expression becomes a perfect square by adding a perfect square multiple of $x^2y^2$". Since the motive is to factor the trinomial, it might be said that if $k<2$, the trinomial can be expressed as the product of two real factors by the method of the difference of two squares. To do this, add and subtract $(2-k)\ x^2y^2$.

The directions for solving a system of equations in several unknowns do not express what is meant (p. 64). They are "1. Eliminate one of the unknowns from a pair of equations, thus reducing the system by one equation and one unknown. 2. Continue this process until the value of one unknown is found. 3. Find the remaining unknowns by substituting back in the preceding equations." If the system contains more than two equations, one of the unknowns will have to be eliminated from more than one pair before the system is effectively reduced. The student who overlooks these directions and follows the worked example will escape the confusion.

The chapter on theory of equations may enable a diligent student to solve some numerical equations; but it has many imperfections. The section on transformations is little more than a set of rules for manipulating coefficients. The following quotation (p. 100) should have been edited: "All cubic equations in $x$ will have a graph beginning with a negative $y$-value and ending with a positive value for $y$ if the first coefficient (that is of $x^3$) is positive." In a true sense of the word, the graph of a cubid equation in $x$ consists of three vertical straight lines, or one such line, according as the equation has three real roots or one. The graph exhibited in the book is the graph of a cubic polynomial. Such a graph neither begins nor ends.

The usable features of the book probably outweigh its unfortunate defects. In the hands of a comptetnt instructor, it may be of great assistance to the war effort.

*Central Y. M. C. A. College, Chicago.* G. D. GORE.

*Algebra. A Textbook of Determinants, Matrices and Algebraic Forms.* By W. L Ferrar. The Clarendon Press, Oxford. 1941. vi+202 pages. $3.50.

According to the preface, it was the author's intention to present in this book material "which might reasonably be reckoned as an essential part of an undergraduate's education" but to exclude "topics appropriate to post-graduate or to highly specialized courses of study". In the opinion of the reviewer the material has been happily chosen and charmingly presented.

Part I is devoted to the theory of determinants, and procedes in a leisurely, careful manner (being aimed at approximately sophomore level) without sacrificing any of the standard theorems on determinants of order $n$. The presentation is not merely excellent, it exhibits a good deal of originality. Noteworthy, because they are unexpected, are brief sections on alternates, and the differentiation of determinants, and a short chapter on symmetric and skew-symmetric determinants, with mention of Pfaffians.

Part II deals with matrices, and includes, for example, linear equations and the characteristic equation. The rank of a matrix is defined in the classical manner, in terms of minors, but the approach to this concept through equivalence of matrices is also given in a final chapter.

In Part III, on linear and quadratic forms, invariants, and covariants, the tempo of the book is considerably accelerated. At two points in the chapter on orthogonal transformations the reviewer feels that the treatment is too brief. The material of this chapter—the expression of an orthogonal matrix in terms of a skew-symmetric matrix—is well worth a more detailed exposition.

Elsewhere in Part III the presentation is uniformly good. Topics treated include Hermitian forms, positive definite real forms and simultaneous reduction of two real quadratic forms. The concluding chapter introduces the theory of invariants, using different processes but making no appeal to the symbolic notation or to representation theory.

Professor Ferrar's book should prove attractive to any student of *higher algebra*, but it ought to be a delight to the engineer or applied mathematician who happens to feel a need for matrices and determinants but doesn't care for the trappings of modern algebra. Little emphasis is placed on the concept of a *field* (in fact most of the proofs make little use of the nature of the underlying *field*) and the notation of a *group* is purposely omitted. References to more advanced texts are given at strategic points, and there are about two hundred exercises of various degrees of difficulty.

*University of Wisconsin.* R. H. BRUCK.

*Algebra for College Students.* By Edwin R. Smith. The Dryden Press, New York, N. Y., 1942. xiv+392 pages. $2.20.

This is a good textbook. It contains an adequate introductory chapter that reviews high school algebra. Then follow the further chapters of customary *College Algebra* and a chapter on *Statistics*, which is hardly customary in books of this type. In each chapter, every topic is well presented by brief and clear explanations in connection with completely solved illustrative examples, which are followed by a large number of well chosen, graded exercises, some of which relate to every-day life. The so-called *story problems*, for which the student is to set up, and solve equations, are very commendable; there are many of these and they are well chosen; a teacher will not wish for a greater variety of problems than this book offers.

This is a conservative text. The material covered, as stated above, is about that which is customary, and the author makes no attempt to introduce the elementary notion of the derivative, of the calculus, or to present any other topic of an advanced or unusual nature.

The book has an attractive cover. The print is clear, and the arrangement of paragraphs and developments is orderly. It is our conviction that the mechanical make-up of the book is such that it will win the student's favor before he studies it.

Because of the clearness of the explanations and the comprehensive scope of this book, we consider it a very appropriate text at the present (since time must now not be wasted on non-essential material).

*Manchester College.* J. E. DOTTERER.

*A Treatise on Projective Differential Geometry.* By Ernest Preston Lane, The University of Chicago Press, 1942. ix+466 pages. $6.00.

Since the appearance ten years ago of the author's *Projective Differential Geometry of Curves and Surfaces*, the University of Chicago Press (1932), much has been added to the methods and materials of the subject of *Projective Differential Geometry*. The present treatise has the aim of giving a connected exposition of the theory to date. Any critic who knows the subject may find many of his pet ideas missing from the treatment. However, no favors are shown—if a topic lends itself to the main development, it is included, if it does not it is either omitted or relegated to the list of supplementary theorems. Even some of the materials of the former volume are omitted.

The present reviewer has used the earlier volume as a text since its appearance. A common complaint of the students was the pages of omissions between the lines of the text. Such omissions seem to be necessary in the subject of differential geometry. As one mathematician has well said, the two main difficulties of the subject are: first, introducing a coordinate system into the subject, and second, eliminating it. Such a procedure of course involves much computation which it is needless to reproduce in a treatment of this kind. In the present volume, more attention is given to the methods and some to the details of carrying them out. Even though the volume under review is a treatise, I am sure it will be a more teachable text than the former volume.

Considerable expansion of certain topics over that found in the first text may be found. For example the work of Wilczynski on the study of curves by means of linear differential equations receives considerably more attention. On the other hand Fubini's use of differential forms in the theory of surfaces is omitted. Three times as much space is devoted to the theory of curves, approximately the same ratio for the theory of surfaces, and about twice as much space is devoted to conjugate nets. Considerably more detail is given to conjugate nets in hyperspace. Plane nets receives and deserves a full twenty-page chapter. Considerable factual material is given in the supplementary theorems placed at appropriate places in the text.

The excellent introductory remarks placed at the beginning of chapters in the earlier text are omitted. This is no doubt fitting for a treatise. In a text for study, however, they found for the student an excellent indication of things to come. Those remarks will be missed by the teacher who may choose this later volume for class room use. The contents of the treatise by chapters may be summarized as follows:

I. *Curves in Hyperspace*—fundamental definitions, differential equation of a curve, invariants and covariants, canonical forms, osculating linear spaces, the principle of duality, power series expansionss in canonical form

II. *Plane Curves*—power series expansions, inflexion and sextactic points, covariant loci and envelopes, differential equations, contact of two curves.

III. *Space Curves*—power series expansions, osculating linear complex, curves with $k$-order contact, intersecting curves with distinct tangents, covariant loci and envelopes.

IV. *Surfaces*—Curves on a surface in hyperspace, nets on a surface, developables and other ruled surfaces, integral surfaces of second-order differential equations.

V, VI. *Surfaces in Ordinary Space*—canonical forms of the differential equations, power series expansions, ruled surfaces, the quadrics of Darboux and Lie, asymptotic osculating quadrics of a curve, various covariant reciprocal lines, the role of duality, hypergeodesics, pangeodesics, quadrics of Moutard, plane and cone curves, differentiation of local coordinates.

VII. *Conjugate Nets in Hyperspace*—discussion of equations of Laplace, transformation of Laplace, terminating, periodic and polar sequences of Laplace, conjugate nets conjugate or harmonic to a given congruence and congruences conjugate or harmonic to a given conjugate net.

VIII. *Conjugate Nets in Ordinary Space*—transformations on, and invariants, covariants and canonical forms of differential equations, axis and ray congruences, special classes of conjugate nets, pencils of conjugate nets, the associate conjugate net, canonical points lines, plane and quadric.

IX. *Plane Nets*—discussion of plane nets using such ideas of surfaces and conjugate nets as carry over to the plane, projection of the asymptotic curves on a plane.

X. *Transformations of Surfaces*—the transformation of Levy, fundamental transformation, transformations of Koenig's Ribaucour, and of Klein-Segre; pairs of surfaces, $W$ congruences.

XI. *Surfaces and Varieties in Hyperspace*—the spaces of immersion of various types of linear osculants of curves on ruled surfaces, quasi-asymptotic curves on ruled surfaces, hyperplane sections of a surface, surface $F$ and of Veronese, varieties as loci of $h$-parameter linear spaces, varieties as envelopes of hyperplanes, varieties as loci of linear spaces in correspondence.

Considerable of the author's previously unpublished work here appears for the first time.

The pages of the treatise are pleasing to the eye; the reviewer is sure that anyone desiring a comprehensive, satisfying book for reference or for teaching the subject will find plenty to satisfy him.

*Michigan State College.* VERNON G. GROVE.